Supplemental Information:
Biased Assimilation, Homophily, and the Dynamics of Polarization

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We provide supplemental information for the article "Biased Assimilation, Homophily, and the Dynamics of Polarization" submitted to Proceedings of the National Academy of Sciences. This document contains proofs of the theorems stated in the paper. Additionally, we state and prove a less restrictive version of Theorem 3 showing that in two-island networks with *non-homogeneous* opinions, if the initial opinions are sufficiently far apart and if $b \geq 1$, the biased opinion formation process produces polarization.

1 Proof of Theorem 1

Recall that

$$x(t+1) := \frac{wx(t) + (x(t))^b s}{w + (x(t))^b s + (1 - x(t))^b (1-s)}$$

Equivalently,

$$\frac{x(t+1)}{1-x(t+1)} = \frac{wx(t) + (x(t))^b s}{w(1-x(t)) + (1-x(t))^b (1-s)} = \frac{w + (x(t))^{b-1} s}{w + (1-x(t))^{b-1}(1-s)} \frac{x(t)}{1-x(t)} \qquad (1.1)$$

First we will show that if $x(t) = \hat{x}$, then for all $t' > t, x(t') = \hat{x}$.

Lemma 1.1. *Assume $b \neq 1$. Fix $t \geq 0$. Let $x(t) = \hat{x}$. Then for all $t' > t, x(t') = \hat{x}$.*

Proof. To prove the lemma, it suffices to show that $x(t+1) = x(t) = \hat{x}$. Recall that

$$\hat{x} := \frac{s^{1/(1-b)}}{s^{1/(1-b)} + (1-s)^{1/(1-b)}}$$

Or equivalently,

$$\left(\frac{\hat{x}}{1-\hat{x}}\right)^{1-b} = \frac{s}{1-s}$$

This implies that when $x(t) = \hat{x}$, $x(t)^{b-1} s = (1-x(t))^{b-1}(1-s)$. Substituting this in (1.1), we get that

$$\frac{x(t+1)}{1-x(t+1)} = \frac{x(t)}{1-x(t)}$$

Or equivalently, $x(t+1) = x(t)$. □

Next we will show that when $b > 1$, $\hat{x}$ is an unstable equilibrium.

Lemma 1.2. *Let $b > 1$. Fix $t \geq 0$.*

1. *If $x(t) > \hat{x}$, then $x(t+1) > x(t)$.*

2. *If $x(t) < \hat{x}$, then $x(t+1) < x(t)$.*

Proof. Again, recall that

$$\left(\frac{\hat{x}}{1-\hat{x}}\right)^{1-b} = \frac{s}{1-s}$$

Therefore, if $x(t) > \hat{x}$, it implies that

$$\frac{x(t)}{1-x(t)} > \frac{\hat{x}}{1-\hat{x}} \Rightarrow \left(\frac{x(t)}{1-x(t)}\right)^{1-b} < \left(\frac{\hat{x}}{1-\hat{x}}\right)^{1-b} = \frac{s}{1-s} \text{ (since } b > 1)$$

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Or equivalently, $(x(t))^{b-1}s > (1 - x(t))^{b-1}(1 - s)$. Substituting this in (1.1), we get that

$$\frac{x(t+1)}{1 - x(t+1)} > \frac{x(t)}{1 - x(t)}$$

Or equivalently, $x(t+1) > x(t)$.

By a similar argument, if $x(t) < \hat{x}$, then $(x(t))^{b-1}s < (1 - x(t))^{b-1}(1 - s)$. Again, substituting this in (1.1), we get that

$$\frac{x(t+1)}{1 - x(t+1)} < \frac{x(t)}{1 - x(t)}$$

Or equivalently, $x(t+1) < x(t)$. □

Next we will show that when $b > 1$, either $\lim_{t \to \infty} x(t) = 1$ or $\lim_{t \to \infty} x(t) = 0$.

Lemma 1.3. *Let $b > 1$. Fix $t \geq 0$.*

1. *If $x(t) > \hat{x}$, then $\lim_{t \to \infty} x(t) = 1$.*

2. *If $x(t) < \hat{x}$, then $\lim_{t \to \infty} x(t) = 0$.*

Proof. For the proof, we will assume that $x(t) > \hat{x}$ and show that $\lim_{t \to \infty} x(t) = 1$. The case when $x(t) < \hat{x}$ can be argued in an analogous way.

By definition, we know that for all $t \geq 0, x(t) \in [0,1]$. Further, from Lemma 1.2, we know that the sequence $\{x(t')_{t' \geq t}\}$ is strictly increasing. Since the sequence is strictly increasing and bounded, it must converge either to 1 or to some value in the interval $[x(t), 1)$. Consider the function $g : [0,1] \to \mathbb{R}$ defined as

$$g(y) := \frac{w + y^b s}{w + y^b s + (1 - y)^b (1 - s)} - y$$

Observe that for all $t \geq 0$, $x(t+1) - x(t) = g(x(t))$. Therefore,

(a) for all $y \in [x(t), 1)$, $g(y) > 0$ (since, by Lemma 1.2, the sequence $\{x(t')_{t' \to t}\}$ is strictly increasing), and

(b) $g(1) = 0$.

For the purpose of contradiction, assume that $\lim_{t \to \infty} x(t) = a$, where $x(t) \leq a < 1$. This implies, for every $\epsilon > 0$, there exists a $t(\epsilon)$ such that for all $t' \geq t(\epsilon)$, $x(t' + 1) - x(t') < \epsilon$, or equivalently, that for all $t' \geq t(\epsilon)$, $g(x(t')) < \epsilon$.

Let $\min_{y \in [x(t),a]} g(y) = c$. It implies for all $y \in [x(t), a]$, $g(y) \geq c$. From (a), it follows that $c > 0$. Setting $\epsilon = c$, our analysis implies the following two properties of g: (1) for all $t \geq 0, g(x(t)) \geq c$, and (2) for all $t' \geq t(\epsilon), g(x(t')) < c$, which contradict each other. This completes the proof by contradiction. □

Using a similar argument we can show that when $b < 1$, $\hat{x}$ is a stable equilibrium.

Lemma 1.4. *Let $b < 1$. Fix $t \geq 0$.*

1. *If $x(t) > \hat{x}$, then $x(t+1) < x(t)$.*

2. *If $x(t) < \hat{x}$, then $x(t+1) > x(t)$.*

Lemma 1.5. *Let $b < 1$. Then, $\lim_{t \to \infty} x(t) = \hat{x}$.*

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2   Proof of Theorem 2

Recall that since $b_i = 0$, the opinion of node i at time $t + 1$ is given by

$$x_i(t+1) = \frac{w_{ii}x_i(t) + \sum_{j \in N(i)} w_{ij}x_j(t)}{w_{ii} + d_i} \tag{2.1}$$

where recall that $d_i := \sum_{j \in N(i)} w_{ij}$ is the weighted degree of node i. Let L_G be the weighted laplacian matrix of G. Recall that L_G is given by

$$(L_G)_{ij} = \begin{cases} d_i, & \text{if } i = j \\ -w_{ij}, & \text{if } (i,j) \in E \\ 0, & \text{otherwise} \end{cases}$$

Now consider the vector $L_G\mathbf{x}(t)$. The ith entry of the vector is given by

$$(L_G\mathbf{x}(t))_i = d_ix_i(t) - \sum_{j \in N(i)} w_{ij}x_j(t) = d_ix_i(t) + w_{ii}x_i(t) - \left(w_{ii}x_i(t) + \sum_{j \in N(i)} w_{ij}x_j(t)\right)$$

$$= (d_i + w_{ii})(x_i(t) - x_i(t+1)) \text{ (from (2.1))}$$

Equivalently, in matrix notation,

$$\mathbf{x}(t+1) = (I - DL_G)\mathbf{x}(t) \tag{2.2}$$

where, D is a diagonal matrix such that $D_{ii} = 1/(d_i + w_{ii})$. Note that since G is connected, $d_i > 0$, and therefore D_{ii} is finite. Consider the difference $\eta(G, \mathbf{x}(t+1)) - \eta(G, \mathbf{x}(t))$. Observe that for a vector $\mathbf{y} \in [0,1]^n$, $\eta(G, \mathbf{y}) = \mathbf{y}^\top L_G \mathbf{y}$. Therefore, we have that

$$\begin{aligned}
\eta(G, \mathbf{x}(t+1)) - \eta(G, \mathbf{x}(t)) &= (\mathbf{x}(t+1))^\top L_G(\mathbf{x}(t+1)) - (\mathbf{x}(t))^\top L_G\mathbf{x}(t) \\
&= (\mathbf{x}(t))^\top (I - DL_G)^\top L_G(I - DL_G)\mathbf{x}(t) - (\mathbf{x}(t))^\top L_G\mathbf{x}(t) \text{ (from (2.2))} \\
&= (\mathbf{x}(t))^\top \left((L_G - L_GDL_G)(I - DL_G) - L_G\right)\mathbf{x}(t) \text{ (since } L_G \text{ is symmetric)} \\
&= (\mathbf{x}(t))^\top \left(L_G - L_GDL_G - L_GDL_G - L_GDL_GDL_G - L_G\right)\mathbf{x}(t) \\
&= (\mathbf{x}(t))^\top \left(L_GDL_GDL_G - 2L_GDL_G\right)\mathbf{x}(t) \\
&= (\mathbf{x}(t))^\top L_G^\top D^{1/2}((D^{1/2}L_GD^{1/2} - 2I))D^{1/2}L_G\mathbf{x}(t) \text{ (since } L_G \text{ is symmetric)} \\
&= \mathbf{y}^\top (D^{1/2}L_GD^{1/2} - 2I)\mathbf{y} \text{ (where } \mathbf{y} := D^{1/2}L_G\mathbf{x}(t))
\end{aligned}$$

Thus, in order to show that $\eta(G, \mathbf{x}(t+1)) - \eta(G, \mathbf{x}(t)) \le 0$, it suffices to show that for all vectors $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y}^\top D^{1/2}L_GD^{1/2}\mathbf{y} \le 2||\mathbf{y}||_2^2$. We prove this as Lemma 2.1.

Lemma 2.1. *Consider an arbitrary weighted undirected graph $G = (V, E, w)$ over n nodes. Let L_G be the weighted laplacian matrix of G. Let D be an $n \times n$ diagonal matrix such that for $i = 1, \ldots, n$, $D_{ii} = 1/(d_i + w_{ii})$, where $d_i = \sum_{j \in N(i)} w_{ij}$ is the weighted degree of i in G. Let $\mathbf{y} \in \mathbb{R}^n$ be an arbitrary vector. Then, $\mathbf{y}^\top D^{1/2}L_GD^{1/2}\mathbf{y} \le 2||\mathbf{y}||_2^2$.*

Proof. For $i = 1, \ldots, n$, let $r_i := d_i + w_{ii}$. Let $P := D^{1/2}L_GD^{1/2}$. Then,

$$P_{ij} = \begin{cases} \frac{d_i}{r_i}, & i = j \\ \frac{-w_{ij}}{\sqrt{r_ir_j}}, & (i,j) \in E \\ 0, & \text{otherwise} \end{cases}$$

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Then, we have that

$$
\begin{aligned}
\mathbf{y}^\top P \mathbf{y} = \sum_{i,j} P_{ij} y_i y_j &= \sum_{i=1}^{n} P_{ii} y_i^2 + 2 \sum_{(i,j)\in E} P_{ij} y_i y_j = \sum_i \frac{d_i}{r_i} y_i^2 - 2 \sum_{(i,j)\in E} \frac{w_{ij}}{\sqrt{r_i r_j}} y_i y_j \\
&= \sum_i \left(\frac{1}{r_i} y_i^2 \sum_{j\in N(i)} w_{ij} \right) - 2 \sum_{(i,j)\in E} \frac{w_{ij}}{\sqrt{r_i r_j}} y_i y_j \\
&= \sum_{(i,j)\in E} w_{ij} \left(\frac{y_i^2}{r_i} + \frac{y_j^2}{r_j} \right) - 2 \sum_{(i,j)\in E} \frac{w_{ij}}{\sqrt{r_i r_j}} y_i y_j \\
&= \sum_{(i,j)\in E} w_{ij} \left(\frac{y_i}{\sqrt{r_i}} - \frac{y_j}{\sqrt{r_j}} \right)^2 \\
&= - \sum_{(i,j)\in E} w_{ij} \left(\frac{y_i}{\sqrt{r_i}} + \frac{y_j}{\sqrt{r_j}} \right)^2 + 2 \sum_i \frac{d_i}{r_i} y_i^2 \\
&\leq - \sum_{(i,j)\in E} w_{ij} \left(\frac{y_i}{\sqrt{r_i}} + \frac{y_j}{\sqrt{r_j}} \right)^2 + 2 \sum_i y_i^2 \text{ (since } d_i \leq r_i) \\
&\leq 2 \|\mathbf{y}\|_2^2
\end{aligned}
$$

$\square$

3   Proof of Theorem 3

To prove the theorem, we begin by making three simple observations that hold for all $b \geq 0$. The first observation follows directly from the symmetry of nodes in each set V_1 and V_2.

Lemma 3.1. *Consider nodes $i, j \in V$ such that either both $i, j \in V_1$ or both $i, j \in V_2$. Then for all $t \geq 0$, $x_i(t) = x_j(t)$.*

The next observation allows us to focus on only analyzing the equilibrium opinion of nodes in V_1.

Lemma 3.2. *Consider a node $i \in V_1$ and a node $j \in V_2$. Then, for all $t \geq 0$, $x_i(t) = 1 - x_j(t)$.*

Proof of Lemma 3.2. By induction.
Induction hypothesis: Assume that the statement holds for some $t \geq 0$.
Base case: The statement holds for $t = 0$ by assumption in the theorem statement.
We will now show that the statement holds for $t + 1$.

$$
\frac{x_i(t+1)}{1 - x_i(t+1)} = \frac{(x_i(t))^b}{(1 - x_i(t))^b} \frac{s_i(t)}{d_i - s_i(t)} \tag{3.1}
$$

where $d_i = n(p_s + p_d)$ and, by Lemma 3.1, $s_i(t) = n(p_s x_i(t) + p_d x_j(t))$. On the other hand,

$$
\frac{x_j(t+1)}{1 - x_j(t+1)} = \frac{(x_j(t))^b}{(1 - x_j(t))^b} \frac{s_j(t)}{d_j - s_j(t)} \tag{3.2}
$$

where $s_j(t) = n(p_s x_j(t) + p_d x_i(t))$, and $d_j = n(p_s + p_d) = d_i$. By the induction hypothesis, we know that $x_i(t) = 1 - x_j(t)$. It follows that $S_i(t) = d_i - s_j(t)$. Substituting this into (3.1), we get

$$\frac{x_i(t+1)}{1 - x_i(t+1)} = \frac{(x_i(t))^b}{(1 - x_i(t))^b} \frac{s_i(t)}{d_i - s_i(t)} = \frac{(1 - x_j(t))^b}{(x_j(t))^b} \frac{d_j - s_j(t)}{s_j(t)} = \frac{1 - x_j(t+1)}{x_j(t+1)}$$

where the last equality follows from (3.2). It follows that $x_i(t+1) = 1 - x_j(t+1)$.

This completes the inductive proof. $\qquad\square$

Lemma 3.2 implies that if we prove the theorem statement for nodes in V_1, we get the proof for nodes in V_2 for free. So, in the rest of the proof, we only make statements about nodes in V_1. The third observation lower bounds the opinions of nodes in V_1.

Lemma 3.3. *Consider a node $i \in V_1$. For all $t \geq 0$, $x_i(t) \in [\frac{1}{2}, 1]$.*

Proof of Lemma 3.3. It is easy to see that for all $t \geq 0$, $x_i(t) \leq 1$. We will prove that $x_i(t) \geq \frac{1}{2}$ by induction over t.

Base case: The statement holds for $t = 0$ by assumption in the theorem statement.

Induction hypothesis: Assume that the lemma statement holds for some $t \geq 0$, *i.e.*, assume that $x_i(t) \geq \frac{1}{2}$ for some $t \geq 0$.

We will show that the lemma statement holds for $t + 1$.

$$\begin{aligned}
\frac{x_i(t+1)}{1 - x_i(t+1)} &= \frac{(x_i(t))^b}{(1 - x_i(t))^b} \frac{S_i(t)}{d_i - s_i(t)} \\
&\geq \frac{(x_i(t))^b}{(1 - x_i(t))^b} \text{ (since } s_i(t) > d_i - s_i(t)) \\
&\geq 1 \text{ (since } x_i(t) \geq \frac{1}{2} \text{ by the induction hypothesis, and } b \geq 0)
\end{aligned}$$

This implies $x_i(t+1) \geq \frac{1}{2}$, completing the inductive proof. $\qquad\square$

Recall that i's opinion at time $t + 1$ is given by

$$x_i(t+1) = \frac{(x_i(t))^b s_i(t)}{(x_i(t))^b s_i(t) + (1 - x_i(t))^b (d_i - s_i(t))}$$

where $s_i(t) = n(p_s x_i(t) + p_d(1 - x_i(t)))$, and $d_i = n(p_s + p_d)$. Now consider the equation

$$x_i(t+1) = x_i(t) \tag{3.3}$$

We will show that if $b \geq 1$ or $b < \frac{2}{h_G + 1}$, (3.3) has no solution in $(\frac{1}{2}, 1)$, whereas if $1 > b \geq \frac{2}{h_G + 1}$, there exists a unique solution to (3.3) in $(\frac{1}{2}, 1)$.

Lemma 3.4. *Consider a node $i \in V_1$. Fix $t \geq 0$.*

(a) If $b \geq 1$, for every $x_i(t) \in (\frac{1}{2}, 1)$, $x_i(t+1) > x_i(t)$.

(b) If $1 > b \geq \frac{2}{h_G + 1}$, there exists a unique solution, say $\hat{x}$, to Eq.(3.3) in $(\frac{1}{2}, 1)$.

(c) If $b < \frac{2}{h_G + 1}$, for every $x_i(t) \in (\frac{1}{2}, 1)$, $x_i(t+1) < x_i(t)$.

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Proof of Lemma 3.4. Consider the function $f : [0, 1] \to \mathbb{R}$ defined as

$$f(y; b) := \begin{cases} 1, & y \in [0, 1], b = 1 \\ 0, & y \in [0, 1], b = 2 \\ \frac{2}{b} - 1, & y = \frac{1}{2}, b > 0 \\ \frac{(y)^{2-b} - (1-y)^{2-b}}{y(1-y)^{1-b} - y^{1-b}(1-y)}, & \text{otherwise} \end{cases} \tag{3.4}$$

We will first prove a few properties of f and then use those properties to prove Lemma 3.4.

Proposition 3.1. *1. For all $b > 0$, f is continuous over $[0, 1]$.*

2. If $0 < b < 1$, f is strictly increasing over $[\frac{1}{2}, 1]$.

3. If $b \geq 1$, for all $y \in [0, 1)$, $f(y; b) \leq 1$.

Proof. 1. Observe that f is continuous when $b = 1$ or $b = 2$. So, we only need to show that f is continuous at $y = \frac{1}{2}$ when $b \neq 1$ and $b \neq 2$. Let $p(y; b) := (y)^{2-b} - (1 - y)^{2-b}$ and $q(y; b) := y(1 - y)^{1-b} - y^{1-b}(1 - y)$. Observe that when $b \neq 1$ and $b \neq 2$, both p and q are differentiable on $[0, 1]$. For $y \in [0, 1]$,

$$p'(y; b) = (2-b)(y^{1-b} + (1-y)^{1-b}); \quad q'(y; b) = (1-y)^{1-b} - (1-b)y(1-y)^{-b} - (1-b)y^{-b}(1-y) + y^{1-b}$$

Therefore,

$$\lim_{y \to 1/2} \frac{p'(y; b)}{q'(y; b)} = \lim_{y \to 1/2} \frac{(2 - b)(y^{1-b} + (1 - y)^{1-b})}{(1 - y)^{1-b} - (1 - b)y(1 - y)^{-b} - (1 - b)y^{-b}(1 - y) + y^{1-b}} = \frac{2}{b} - 1 \tag{3.5}$$

So, we have that

$$\lim_{y \to 1/2} f(y; b) = \lim_{y \to 1/2} \frac{p(y; b)}{q(y; b)} = \lim_{y \to 1/2} \frac{p'(y)}{q'(y)} \text{ (using L'Hôpital's rule)} = \frac{2}{b} - 1 \text{ (from (3.5))} = f(\frac{1}{2}; b)$$

Therefore, when $b \neq 1$ and $b \neq 2$, f is continuous at $\frac{1}{2}$.

2. Assume $0 < b < 1$. Fix $y_1, y_2 \in [\frac{1}{2}, 1]$ such that $y_1 > y_2$. We will show that $f(y_1; b) > f(y_2; b)$. For conciseness of expression, define $\bar{y}_1 := 1 - y_1$ and $\bar{y}_2 := 1 - y_2$. Then

$$y_1 y_2 - y_1 \bar{y}_2 > (y_1 y_2)^{1-b} - (y_1 \bar{y}_2)^{1-b} \tag{3.6}$$

Similarly,

$$\bar{y}_1 y_2 - \bar{y}_1 \bar{y}_2 > (\bar{y}_1 y_2)^{1-b} - (\bar{y}_1 \bar{y}_2)^{1-b} \tag{3.7}$$

Adding (3.6) and (3.7), we get

$$y_1 y_2 - y_1 \bar{y}_2 + \bar{y}_1 y_2 - \bar{y}_1 \bar{y}_2 > (y_1 y_2)^{1-b} - (y_1 \bar{y}_2)^{1-b} + (\bar{y}_1 y_2)^{1-b} - (\bar{y}_1 \bar{y}_2)^{1-b}$$

Or equivalently,

$$(y_1 y_2 - \bar{y}_1 \bar{y}_2) - \left((y_1 y_2)^{1-b} - (\bar{y}_1 \bar{y}_2)^{1-b}\right) > (y_1 \bar{y}_2 - \bar{y}_1 y_2) - \left((y_1 \bar{y}_2)^{1-b} - (\bar{y}_1 y_2)^{1-b}\right) \tag{3.8}$$

Moreover, since $y_1, y_2 \in [\frac{1}{2}, 1]$ and $y_1 > y_2$,

$$y_1 y_2 - \bar{y}_1 \bar{y}_2 > 0; (y_1 y_2)^{1-b} - (\bar{y}_1 \bar{y}_2)^{1-b} > 0; y_1 \bar{y}_2 - \bar{y}_1 y_2 > 0; (y_1 \bar{y}_2)^{1-b} - (\bar{y}_1 y_2)^{1-b} > 0 \tag{3.9}$$

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(3.8) and (3.9) imply that

$$\frac{y_1 y_2 - \bar{y}_1 \bar{y}_2}{y_1 \bar{y}_2 - \bar{y}_1 y_2} > \frac{(y_1 y_2)^{1-b} - (\bar{y}_1 \bar{y}_2)^{1-b}}{(y_1 \bar{y}_2)^{1-b} - (\bar{y}_1 y_2)^{1-b}}$$

Rearranging, we get

$$\frac{(y_1)^{2-b} - \bar{y}_1^{2-b}}{y_1 \bar{y}_1^{1-b} - y_1^{1-b} \bar{y}_1} = f(y_1; b) > \frac{(y_2)^{2-b} - \bar{y}_2^{2-b}}{y_2 \bar{y}_2^{1-b} - y_2^{1-b} \bar{y}_2} = f(y_2; b)$$

3. Since f is symmetric about $y = \frac{1}{2}$, we will prove the theorem for $y \in [\frac{1}{2}, 1)$. Fix $y \in [\frac{1}{2}, 1)$. Observe that when $b \geq 1$, $(1 - y)^{1-b} \geq y^{1-b}$ (since $y \geq 1 - y$). Equivalently

$$y(1 - y)^{1-b} \geq y^{2-b} \tag{3.10}$$

For the same reason,

$$y^{1-b}(1 - y) \leq (1 - y)^{2-b} \tag{3.11}$$

From (3.10) and (3.11), it follows that

$$y(1 - y)^{1-b} - y^{1-b}(1 - y) \geq (y)^{2-b} - (1 - y)^{2-b}$$

or equivalently, $f(y; b) \leq 1$.

□

Using these properties of f we will prove Lemma 3.4.

1. If $b \geq 1$, then for all $y \in [0, 1)$, $f(y; b) \leq 1$ (by Proposition 3.1) $< h_G$. Therefore, for $y \in [\frac{1}{2}, 1)$,

$$\frac{(y)^{2-b} - (1 - y)^{2-b}}{y(1 - y)^{1-b} - y^{1-b}(1 - y)} < h_G$$

$$\Leftrightarrow y^{2-b} - (1 - y)^{2-b} < h_G(y(1 - y)^{1-b} - y^{1-b}(1 - y))$$

$$\Leftrightarrow y^{2-b} + h_G y^{1-b}(1 - y) < (1 - y)^{2-b} + h_G y(1 - y)^{1-b}$$

$$\Leftrightarrow y^{1-b}(y + (1 - y)h_G) < (1 - y)^{1-b}((1 - y) + h_G y)$$

$$\Leftrightarrow \frac{y}{1 - y} < \left(\frac{y}{1 - y}\right)^b \cdot \frac{(1 - y) + h_G y}{y + (1 - y)h_G}$$

For $y = x_i(t)$, the right hand side of the last inequality above is equal to $x_i(t+1)/(1-x_i(t+1))$, implying that $x_i(t + 1) > x_i(t)$.

2. If $1 > b \geq \frac{2}{h_G+1}$, then observe that $f(\frac{1}{2}; b) = \frac{2}{b} - 1 \leq h_G < f(1; b) = \infty$. Since f is a continuous function (by Proposition 3.1), therefore, by the intermediate value theorem, there must exist a $\hat{y} \in [\frac{1}{2}, 1)$ such that $f(\hat{y}; b) = h_G$. Equivalently,

$$\frac{(\hat{y})^{2-b} - (1 - \hat{y})^{2-b}}{\hat{y}(1 - \hat{y})^{1-b} - \hat{y}^{1-b}(1 - \hat{y})} = h_G$$

Rearranging the above expression, we get

$$\frac{\hat{y}}{1 - \hat{y}} = \left(\frac{\hat{y}}{1 - \hat{y}}\right)^b \cdot \frac{(1 - \hat{y}) + h_G \hat{y}}{\hat{y} + (1 - \hat{y})h_G}$$

Again, for $\hat{y} = x_i(t)$, we have that $x_i(t + 1) = x_i(t)$. The uniqueness of $\hat{x}$ follows from the fact that, by Proposition 3.1, f is strictly increasing over $(\frac{1}{2}, 1]$.

3. If $b < \frac{2}{h_G+1}$, then for all $y \in [\frac{1}{2}, 1]$, $f(y;b) \geq f(\frac{1}{2};b)$ (by Proposition 3.1) $= \frac{2}{b} - 1 > h_G$. In other words,

$$\frac{(y)^{2-b} - (1-y)^{2-b}}{y(1-y)^{1-b} - y^{1-b}(1-y)} > h_G$$

Again, rearranging the above expression, we get

$$\frac{y}{1-y} > \left(\frac{y}{1-y}\right)^b \cdot \frac{(1-y) + h_G y}{y + (1-y)h_G}$$

Again, for $y = x_i(t)$, the right hand side of the last inequality above is equal to $x_i(t+1)$, implying that $x_i(t+1) < x_i(t)$.

This concludes the proof of Lemma 3.4. $\qquad \square$

Next we will prove Theorem 3 for the case of persistent disagreement, the cases of polarization and consensus are limiting cases of that case as $b \to 1$ and $b \to 2/(h_G+1)$ respectively. We will show that when $1 > b \geq \frac{2}{h_G+1}$, the value $\hat{x}$ defined in Lemma 3.4(b) is a stable equilibrium. The other two cases can be formally proven using an argument similar to the one below. Next we will show that when $1 > b \geq \frac{2}{h_G+1}$, the sequence $\{x_i(t)\}$ is bounded.

Lemma 3.5. *Consider a node $i \in V_1$. Let $1 > b \geq \frac{2}{h_G+1}$. Let $\hat{x} \in (\frac{1}{2}, 1)$ be the solution to (3.3).*

1. *If $x_0 < \hat{x}$, then for all $t > 0$, $x_i(t) < \hat{x}$.*

2. *If $x_0 > \hat{x}$, then for all $t > 0$, $x_i(t) > \hat{x}$.*

Proof of Lemma 3.5. We will prove statement (1). Statement (2) can be proven using a similar argument.

Proof by induction.

Induction hypothesis: Assume that the lemma statement holds for some $t \geq 0$, *i.e.*, assume that $x_i(t) < \hat{x}$ for some $t \geq 0$.

Base case: The statement holds for $t = 0$ by assumption.

We will show that the lemma statement holds for $t + 1$.

$$\frac{x_i(t+1)}{1 - x_i(t+1)} = \frac{(x_i(t))^b}{(1-x_i(t))^b} \frac{s_i(t)}{d_i - s_i(t)} < \frac{(\hat{x})^b}{(1-\hat{x})^b} \frac{s_i(t)}{d_i - s_i(t)} \quad \text{(since } \frac{1}{2} < x_i(t) < \hat{x}, \text{ and } b > 0\text{)}$$

Observe that since $x_i(t) < \hat{x}$ and $p_s > p_d$, $s_i(t) = n(p_s x_i(t) + p_d(1 - x_i(t))) < n(p_s \hat{x} + p_d(1 - \hat{x}))$. Therefore,

$$\frac{s_i(t)}{d_i - s_i(t)} < \frac{p_s \hat{x} + p_d(1 - \hat{x})}{p_s(1 - \hat{x}) + p_d \hat{x}}$$

As a result,

$$\frac{x_i(t+1)}{1 - x_i(t+1)} < \frac{(\hat{x})^b}{(1-\hat{x})^b} \frac{p_s \hat{x} + p_d(1-\hat{x})}{p_s(1-\hat{x}) + p_d \hat{x}} = \frac{\hat{x}}{1 - \hat{x}} \quad \text{(by definition of } \hat{x}\text{)}$$

This implies $x_i(t+1) < \hat{x}$. This completes the inductive proof. $\qquad \square$

Next we will prove that when $1 > b \geq \frac{2}{h_G+1}$, the sequence $\{x_i(t)\}$ is monotone.

Lemma 3.6. *Consider a node $i \in V_1$. Let $1 > b \geq \frac{2}{h_G+1}$. Let $\hat{x} \in (\frac{1}{2}, 1)$ be the solution to (3.3).*

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1. If $x_0 < \hat{x}$, the sequence $\{x_i(t)\}$ is strictly increasing.

2. If $x_0 > \hat{x}$, the sequence $\{x_i(t)\}$ is strictly decreasing.

Proof of Lemma 3.6. We will prove statement (1); statement (2) can be proven using a similar argument.

Assume $x_0 < \hat{x}$. Then, from Lemma 3.5, we know that for all $t \geq 0, x_i(t) < \hat{x}$. Fix $t \geq 0$. Let $x_i(t) = y < \hat{x}$. Recall that by definition of $\hat{x}$, if $x_i(t) = \hat{x}$, $x_i(t+1) = x_i(t)$. Equivalently, $f(\hat{x}; b) = h_G$, where f is defined by (3.4). From Proposition 3.1, we know that f is strictly increasing over the interval $(\frac{1}{2}, \hat{x})$. Therefore, $f(y; b) < f(\hat{x}; b) = h_G$. Equivalently,

$$\frac{(y)^{2-b} - (1-y)^{2-b}}{y(1-y)^{1-b} - y^{1-b}(1-y)} < h_G$$

Rearranging, we get

$$\frac{y}{1-y} < \left(\frac{y}{1-y}\right)^b \cdot \frac{(1-y) + h_G y}{y + (1-y)h_G} = \frac{x_i(t+1)}{1 - x_i(t+1)}$$

Equivalently, $x_i(t+1) > x_i(t)$. □

Using the fact that the sequence $\{x_i(t)\}$ is monotone and bounded, next we will prove that it converges to $\hat{x}$.

Lemma 3.7. *Consider a node $i \in V_1$. Let $1 > b \geq \frac{2}{h_G + 1}$. Let $\hat{x} \in (\frac{1}{2}, 1)$ be the solution to (3.3). Then, $\lim_{t \to \infty} x_i(t) = \hat{x}$.*

Proof. For the proof, we will assume that the initial opinion $x_i(0) = x_0 \leq \hat{x}$. The case when $x_0 > \hat{x}$ can be argued in an analogous way.

Observe that if $x_0 = \hat{x}$, then by Lemma 3.4, it follows that for all $t \geq 0$, $x_i(t+1) = \hat{x}$, and we are done. So let us assume that $\frac{1}{2} < x_0 < \hat{x}$. From Lemma 3.5 and Lemma 3.6, we know that the sequence $\{x_i(t)\}$ is strictly increasing and bounded. This implies that the sequence must converge either to $\hat{x}$ or to some value in the interval $[x_0, \hat{x})$. Consider the function $g : [0, 1] \to \mathbb{R}$ defined as

$$g(y) := \frac{y^b(h_G y + (1-y))}{y^b(h_G y + (1-y)) + (1-y)^b(h_G(1-y) + y)} - y$$

Observe that for all $t \geq 0$, $x_i(t+1) - x_i(t) = g(x_i(t))$. Therefore,

(a) for all $y \in (\frac{1}{2}, \hat{x})$, $g(y) > 0$ (since, by Lemma 3.6, the sequence $\{x_i(t)\}$ is strictly increasing), and

(b) $g(\hat{x}) = 0$ (by definition of $\hat{x}$).

For the purpose of contradiction, assume that $\lim_{t \to \infty} x_i(t) = a$, where $x_0 \leq a < \hat{x}$. This implies, for every $\epsilon > 0$, there exists a $t(\epsilon)$ such that for all $t \geq t(\epsilon)$, $x_i(t+1) - x_i(t) < \epsilon$, or equivalently, that for all $t \geq t(\epsilon)$, $g(x_i(t)) < \epsilon$.

Let $\min_{y \in [x_0, a]} g(y) = c$. It implies for all $y \in [x_0, a]$, $g(y) \geq c$. From (a), it follows that $c > 0$. Setting $\epsilon = c$, our analysis implies the following two properties of g: (1) for all $t \geq 0, g(x_i(t)) \geq c$, and (2) for all $t \geq t(\epsilon), g(x_i(t)) < c$, which contradict each other. This completes the proof by contradiction. □

This completes the proof of Theorem 3.

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4   Two-island Networks with Non-homogeneous Opinions

In this section, we prove a less restrictive version of the polarization result in Theorem 3, which does not require that the initial opinions in each island be homogeneous. We show that in a two-island network, if the bias parameter $b \geq 1$ and the initial opinions of the two islands are sufficiently far apart relative to the homophily index h_G, then the biased opinion formation process results in polarization.

Theorem 4.1. *Let $G = (V_1, V_2, E, w)$ be a (n, n, p_s, p_d)-two island network. For all $(i,j) \in E$, let $w_{ij} = 1$. Fix $\epsilon \in (0, \frac{1}{2}]$. Assume for all $i \in V_1$, $x_i(0) \geq \frac{1}{2} + \epsilon$ and for all $i \in V_2$, $x_i(0) \leq \frac{1}{2} - \epsilon$. Assume for all $i \in V$, the bias parameter $b_i = b \geq 1$. Then, if $\epsilon > \frac{1}{2h_G}$, for all $i \in V_1$, $\lim_{t \to \infty} x_i(t) = 1$, and for all $i \in V_2$, $\lim_{t \to \infty} x_i(t) = 0$.*

Proof. We will show that the opinions of individuals in V_1 are strictly increasing whereas that of individuals in V_2 are strictly decreasing.

Lemma 4.1. *Fix $t \geq 0$. Then,*

1. *For all $i \in V_1$, if $x_i(t) \in [\frac{1}{2} + \epsilon, 1)$, then $x_i(t+1) > x_i(t)$.*

2. *For all $i \in V_2$, if $x_i(t) \in (0, \frac{1}{2} - \epsilon]$, then $x_i(t+1) < x_i(t)$.*

Proof. We will prove Statement 1 of the lemma. Statement 2 can be proven using an analogous argument. Fix an individual $i \in V_1$.

$$\frac{x_i(t+1)}{1 - x_i(t+1)} = \frac{w_{ii} x_i(t) + x_i(t)^b s_i(t)}{w_{ii}(1 - x_i(t)) + (1 - x_i(t))^b (d_i - s_i(t))}$$

$$= \frac{w_{ii} x_i(t) + x_i(t)^b \left(\sum_{j \in N(i) \cap V_1} x_j(t) + \sum_{j \in N(i) \cap V_2} x_j(t) \right)}{w_{ii}(1 - x_i(t)) + (1 - x_i(t))^b \left(\sum_{j \in N(i) \cap V_1} (1 - x_j(t)) + \sum_{j \in N(i) \cap V_2} (1 - x_j(t)) \right)}$$

Observe that $\sum_{j \in N(i) \cap V_1} x_j(t) \geq np_s \left(\frac{1}{2} + \epsilon \right)$ and $\sum_{j \in N(i) \cap V_2} (1 - x_j(t)) \leq np_d$. Therefore,

$$\frac{x_i(t+1)}{1 - x_i(t+1)} \geq \frac{w_{ii} x_i(t) + x_i(t)^b \left(np_s \left(\frac{1}{2} + \epsilon \right) + 0 \right)}{w_{ii}(1 - x_i(t)) + (1 - x_i(t))^b \left(np_s \left(\frac{1}{2} - \epsilon \right) + np_d \right)}$$

$$= \frac{w_{ii} x_i(t) + x_i(t)^b \left(\frac{1}{2} + \epsilon \right)}{w_{ii}(1 - x_i(t)) + (1 - x_i(t))^b \left(\frac{1}{2} - \epsilon + \frac{1}{h_G} \right)}$$

$$> \frac{w_{ii} x_i(t) + x_i(t)^b}{w_{ii}(1 - x_i(t)) + (1 - x_i(t))^b} \ \text{(since } \epsilon > \frac{1}{2h_G} \text{)}$$

$$> \frac{x_i(t)}{1 - x_i(t)} \ \text{(since } x_i(t) > \frac{1}{2} \text{ and } b \geq 1 \text{)}$$

Or equivalently, $x_i(t+1) > x_i(t)$. $\qquad \square$

Next we will show that for an individual $i \in V_1$, $x_i(t) \in [\frac{1}{2} + \epsilon, 1]$ for all $t \geq 0$, and for an individual $i \in V_2$, $x_i(t) \in [0, \frac{1}{2} - \epsilon]$ for all $t \geq 0$.

Lemma 4.2.   1. *Fix individual $i \in V_1$. For all $t \geq 0$, $x_i(t) \in [\frac{1}{2} + \epsilon, 1]$.*

2. *Fix individual $i \in V_2$. For all $t \geq 0$, $x_i(t) \in [0, \frac{1}{2} - \epsilon]$.*

Proof. We will prove Statement 1 of the lemma. Statement 2 can be proven using an analogous argument. Proof by induction on t.

Base case: The statement holds for $t = 0$ by assumption.

Induction hypothesis: Assume that the statement holds for some $t \geq 0$.

We will show that the statement holds for $t + 1$. If $x_i(t) = 1$, then $x_i(t') = 1$ for all $t' \geq t$, and we are done. So let us assume $x_i(t) < 1$. Then, by Lemma 4.1, $x_i(t + 1) > x_i(t)$. Therefore, $x_i(t + 1) \in [\frac{1}{2} + \epsilon, 1]$. Therefore, the statement holds for $t + 1$. This concludes the proof by induction. $\square$

Next we will show that for an individual $i \in V_1$, $\lim_{t \to \infty} x_i(t) = 1$. The corresponding statement for individuals in V_2 can be proven using an analogous argument.

Lemma 4.3. *Fix an individual $i \in V_1$. Then, $\lim_{t \to \infty} x_i(t) = 1$.*

Proof. The proof is along the same lines as that for Lemma 3.7. Again, observe that if $x_i(t) = 1$, then for all $t' \geq t$, $x_i(t') = 1$, and we are done. Define a function $g : [\frac{1}{2} + \epsilon, 1] \to \mathbb{R}$, as follows:

$$g(y) := \frac{w_{ii} y + y^b \left(\frac{1}{2} + \epsilon \right)}{w_{ii} + y^b \left(\frac{1}{2} + \epsilon \right) + (1 - y)^b \left(\frac{1}{2} - \epsilon + \frac{1}{h_G} \right)} - y$$

Observe that for all $t \geq 0$, for all $x_i(t) \in [\frac{1}{2} + \epsilon, 1)$, $x_i(t + 1) - x_i(t) \geq g(x_i(t)) > 0$. Moreover, $g(1) = 0$. For the purpose of contradiction, assume that $\lim_{t \to \infty} x_i(t) = a$, where $\frac{1}{2} + \epsilon \leq a < 1$. This implies, for every $\delta > 0$, there exists a $t(\delta)$ such that for all $t \geq t(\delta)$, $x_i(t + 1) - x_i(t) < \delta$, which implies that for all $t \geq t(\delta)$, $g(x_i(t)) < \delta$.

Let $\min_{y \in [\frac{1}{2} + \epsilon, a]} g(y) = c$. It implies for all $y \in [\frac{1}{2} + \epsilon, a]$, $g(y) \geq c$. Since $g(y) > 0$ for $y \in [\frac{1}{2} + \epsilon, 1)$, it follows that $c > 0$. Setting $\delta = c$, our analysis implies the following two properties of g: (1) for all $t \geq 0$, $g(x_i(t)) \geq c$, and (2) for all $t \geq t(\delta)$, $g(x_i(t)) < c$, which contradict each other. This completes the proof by contradiction. $\square$

$\square$

5 Proof of Theorem 4

Let $|S(t)| = k$. Then, the opinion update under the flocking process can be written in matrix form as

$$\mathbf{x}(t + 1) = (1 - \epsilon)\mathbf{x}(t) + \epsilon A(t)\mathbf{x}(t)$$

where $A(t)$ is a $n \times n$ matrix given by

$$A_{ij}(t) = \begin{cases} \frac{1}{k}, & \text{if } i \in S(t), j \in S(t) \\ 1, & \text{if } i = j \text{ and } i \notin S(t) \\ 0, & \text{otherwise} \end{cases}$$

Observe that $A(t)$ is doubly-stochastic. Then

$$\begin{aligned}
\gamma(\mathbf{x}(t + 1)) &= \gamma((1 - \epsilon)\mathbf{x}(t) + \epsilon A(t)\mathbf{x}(t)) \text{ (by definition of } \mathbf{x}(t + 1)) \\
&\leq (1 - \epsilon)\gamma(\mathbf{x}(t)) + \epsilon\gamma(A(t)\mathbf{x}(t)) \text{ (since } \gamma \text{ is convex in } \mathbf{x}) \\
&\leq (1 - \epsilon)\gamma(\mathbf{x}(t)) + \epsilon\gamma(\mathbf{x}(t)) \text{ (by Proposition 5.1)} \\
&= \gamma(\mathbf{x}(t))
\end{aligned}$$

Proposition 5.1. $\gamma(A(t)\mathbf{x}(t)) \leq \gamma(\mathbf{x}(t))$.

Proof. Let $\mathbf{y} := A(t)\mathbf{x}(t)$. Since $A(t)$ is doubly stochastic, it follows by a famous theorem by Hardy, Littlewood and Polya, that $\mathbf{x}(t)$ majorizes $\mathbf{y}$. Moreover, $\gamma(\mathbf{x})$ is a convex symmetric function. Therefore, it is a Schur-convex function. By definition, a function $f : \mathbb{R}^n \to \mathbb{R}$ is Schur-convex if $f(\mathbf{x}_1) \geq f(\mathbf{x}_2)$ whenever $\mathbf{x}_1$ majorizes $\mathbf{x}_2$. Therefore, $\gamma(\mathbf{y}) \leq \gamma(\mathbf{x}(t))$. □

6 Proofs of Theorems on Recommender Systems and Polarization

In this section we prove Theorem 5 and Theorem 6 from the main paper. Both theorems rely on the following technical lemma that invokes the Strong Law of Large Numbers to show that the random quantities we care about all take their expected values with probability 1 as $n \to \infty$.

Lemma 6.1. *In the limit as $n \to \infty$, with probability 1,*

(a) for all $i \in V_1$, $|N(i)| \to k$,

(b) for all $i \in V_1$, $\sum_{\substack{j_1 \in V_2 \\ j_1 \text{ is RED}}} Z_{ij_1} \to x_i k$,

(c) for all $i \in V_1$, $\sum_{\substack{j_1 \in V_2 \\ j_2 \text{ is BLUE}}} Z_{ij_2} \to (1 - x_i)k$,

(d) for all $j \in V_2$, $|N(j)| \to \frac{mk}{2n}$,

(e) for every pair of RED books $j, j' \in V_2, M_{jj'} = \sum_{i \in V_1} Z_{ij} Z_{ij'} \to \frac{mk^2(\frac{1}{4} + Var(x_1))}{n^2}$,

(f) for every pair of BLUE books $j, j' \in V_2, M_{jj'} = \sum_{i \in V_1} Z_{ij} Z_{ij'} \to \frac{mk^2(\frac{1}{4} + Var(x_1))}{n^2}$, and

(g) for every RED book j and every BLUE book j', $M_{jj'} = \sum_{i \in V_1} Z_{ij} Z_{ij'} \to \frac{mk^2(\frac{1}{4} - Var(x_1))}{n^2}$.

Proof. Recall that as $n \to \infty$, $m = f(n) \to \infty$. So statements (a) through (g) follow from the Strong Law of Large Numbers. □

Using Lemma 6.1, we will first prove Theorem 6.

6.1 Proof of Theorem 6

Lemma 6.2. *In the limit as $n \to \infty$, SimpleSALSA is polarizing with respect to i if and only if i is biased.*

Proof. Assume without loss of generality that $x_i > \frac{1}{2}$.

Let p_r be the probability that SimpleSALSA recommends a RED book. The proof consists of two steps: first we show that $p_r > \frac{1}{2}$ and $p_r \leq x_i$, and then we show that if $p_r > \frac{1}{2}$ and $p_r \leq x_i$,

SimpleSALSA is polarizing with respect to i if and only if i is biased.

$$p_r = \sum_{\substack{j \in V_2 : j_2 \text{ is RED}}} \mathbb{P}[i \xrightarrow{3} j]$$

$$= \sum_{\substack{j_1 \in N(i) \\ j_1 \text{ is RED}}} \mathbb{P}[i \xrightarrow{1} j_1] \sum_{\substack{j \in V_2 \\ j \text{ is RED}}} \mathbb{P}[j_1 \xrightarrow{2} j] + \sum_{\substack{j_2 \in N(i) \\ j_2 \text{ is BLUE}}} \mathbb{P}[i \xrightarrow{1} j_2] \sum_{\substack{j \in V_2 \\ j \text{ is RED}}} \mathbb{P}[j_2 \xrightarrow{2} j]$$

$$= \sum_{\substack{j_1 \in N(i) \\ j_1 \text{ is RED}}} \frac{1}{|N(i)|} \sum_{\substack{j \in V_2 \\ j \text{ is RED}}} \mathbb{P}[j_1 \xrightarrow{2} j] + \sum_{\substack{j_2 \in N(i) \\ j_2 \text{ is BLUE}}} \frac{1}{|N(i)|} \sum_{\substack{j \in V_2 \\ j \text{ is RED}}} \mathbb{P}[j_2 \xrightarrow{2} j]$$

$$= \sum_{\substack{j_1 \in V_2 \\ j_1 \text{ is RED}}} \frac{Z_{ij_1}}{|N(i)|} \sum_{\substack{j \in V_2 \\ j \text{ is RED}}} \mathbb{P}[j_1 \xrightarrow{2} j] + \sum_{\substack{j_2 \in V_2 \\ j_2 \text{ is BLUE}}} \frac{Z_{ij_2}}{|N(i)|} \sum_{\substack{j \in V_2 \\ j \text{ is RED}}} \mathbb{P}[j_2 \xrightarrow{2} j]$$

$$= \sum_{\substack{j_1 \in V_2 \\ j_1 \text{ is RED}}} \frac{Z_{ij_1}}{|N(i)|} \sum_{\substack{j \in V_2 \\ j \text{ is RED}}} \sum_{i' \in N(j_1) \cap N(j)} \frac{1}{|N(j_1)|} \frac{1}{|N(i')|} + \sum_{\substack{j_2 \in V_2 \\ j_2 \text{ is BLUE}}} \frac{Z_{ij_2}}{|N(i)|} \sum_{\substack{j \in V_2 \\ j \text{ is RED}}} \sum_{i' \in N(j_2) \cap N(j)} \frac{1}{|N(j_2)|} \frac{1}{|N(i')|}$$

$$= \sum_{\substack{j_1 \in V_2 \\ j_1 \text{ is RED}}} \frac{Z_{ij_1}}{|N(i)|} \sum_{\substack{j \in V_2 \\ j \text{ is RED}}} \sum_{i' \in V_1} \frac{Z_{i'j_1} Z_{i'j}}{|N(j_1)||N(i')|} + \sum_{\substack{j_2 \in V_2 \\ j_2 \text{ is BLUE}}} \frac{Z_{ij_2}}{|N(i)|} \sum_{\substack{j \in V_2 \\ j \text{ is RED}}} \sum_{i' \in V_1} \frac{Z_{i'j_2} Z_{i'j}}{|N(j_2)||N(i')|}$$

By Lemma 6.1, in the limit as $n \to \infty$, with probability 1,

$$\sum_{\substack{j_1 \in V_2 \\ j_1 \text{ is RED}}} \frac{Z_{ij_1}}{|N(i)|} \sum_{\substack{j \in V_2 \\ j \text{ is RED}}} \sum_{i' \in V_1} \frac{Z_{i'j_1} Z_{i'j}}{|N(j_1)||N(i')|} \to x_i \frac{1}{k \cdot mk/2n} n \frac{mk^2(\frac{1}{4} + \mathrm{Var}(x_1))}{n^2} = x_i \left(\frac{1}{2} + 2\mathrm{Var}(x_1)\right)$$

and

$$\sum_{\substack{j_2 \in V_2 \\ j_2 \text{ is BLUE}}} \frac{Z_{ij_1}}{|N(i)|} \sum_{\substack{j \in V_2 \\ j \text{ is RED}}} \sum_{i' \in V_1} \frac{Z_{i'j_2} Z_{i'j}}{|N(j_2)||N(i')|} \to (1-x_i) \frac{1}{k \cdot mk/2n} n \frac{mk^2(\frac{1}{4} - \mathrm{Var}(x_1))}{n^2} = (1-x_i) \left(\frac{1}{2} - 2\mathrm{Var}(x_1)\right)$$

Therefore, in the limit as $n \to \infty$, with probability 1,

$$p_r \to x_i \left(\frac{1}{2} + 2\mathrm{Var}(x_1)\right) + (1 - x_i) \left(\frac{1}{2} - 2\mathrm{Var}(x_1)\right)$$

Since $x_i > \frac{1}{2}$ (by assumption), and $\mathrm{Var}(x_1) > 0$ (by assumption), we have that

$$p_r > \frac{1}{2} \text{ and } p_r \le x_i \tag{6.1}$$

First, assume that i is unbiased. Let p be the probability that i accepts the recommendation. Therefore, the probability that the recommended book was RED given that i accepted the recommendation is given by

$$\frac{p_r p}{p_r p + (1 - p_r)p} = p_r \le x_i$$

Therefore, SimpleSALSA is not polarizing.

Now, assume that i is biased. This implies i accepts the recommendation of a RED book with probability x_i and that of a BLUE book with probability $1 - x_i$. Therefore, the probability that the recommended book was RED given that i accepted the recommendation is given by

$$\frac{p_r x_i}{p_r x_i + (1 - x_i)(1 - p_r)} > \frac{p_r x_i}{p_r x_i + p_r(1 - x_i)} \text{ (since } p_r > \frac{1}{2}, \text{ from (6.1))} = x_i$$

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Therefore, by definition, SimpleSALSA is polarizing. Recall that our definition of a biased individual in this section corresponds to $b = 1$. Consider the general case, where i accepts the recommendation of a RED book with probability x_i^b and accepts that of a BLUE book with probability $(1 - x_i)^b$, where $b \geq 0$. Then, the probability that the recommended book was RED given that i accepted the recommendation is given by

$$\frac{p_r x_i^b}{p_r x_i^b + (1 - x_i)^b (1 - p_r)}$$

If $b \geq 1$, then

$$\frac{p_r x_i^b}{p_r x_i^b + (1 - x_i)^b (1 - p_r)} > \frac{p_r x_i}{p_r x_i + (1 - x_i)(1 - p_r)} \quad \left(\text{since } x_i > \frac{1}{2} \text{ and } b \geq 1\right)$$

$$> \frac{p_r x_i}{p_r x_i + p_r(1 - x_i)} \quad \left(\text{since } p_r > \frac{1}{2}, \text{ from (6.1)}\right)$$

$$= x_i$$

This shows that SimpleSALSA is polarizing for any $b \geq 1$.

$\square$

Lemma 6.3. *In the limit as $n \to \infty$ and as $T \to \infty$, SimpleICF is polarizing with respect to i if and only if i is biased.*

Proof. Assume without loss of generality that $x_i > \frac{1}{2}$.

Let p_r be the probability that SimpleICF recommends a RED book. For a node $j \in N(i)$, let $q_{j\text{RED}}$ be the probability that after T two-step random walks starting at j, the node with the largest value of count(j), *i.e.*, j^*, is RED, and $q_{j\text{BLUE}}$ be the corresponding probability that j^* is BLUE. Then,

$$p_r = \sum_{\substack{j_1 \in N(i) \\ j_1 \text{ is RED}}} \mathbb{P}[i \xrightarrow{1} j_1] q_{j_1\text{RED}} + \sum_{\substack{j_2 \in N(i) \\ j_2 \text{ is BLUE}}} \mathbb{P}[i \xrightarrow{1} j_2] q_{j_2\text{RED}}$$

$$= \sum_{\substack{j_1 \in N(i) \\ j_1 \text{ is RED}}} \frac{1}{|N(i)|} q_{j_1\text{RED}} + \sum_{\substack{j_2 \in N(i) \\ j_2 \text{ is BLUE}}} \frac{1}{|N(i)|} q_{j_2\text{RED}}$$

$$= \sum_{\substack{j_1 \in V_2 \\ j_1 \text{ is RED}}} \frac{Z_{ij_1}}{|N(i)|} q_{j_1\text{RED}} + \sum_{\substack{j_2 \in V_2 \\ j_2 \text{ is BLUE}}} \frac{Z_{ij_1}}{|N(i)|} q_{j_2\text{RED}}$$

Consider T two-step random walks starting at a node $j_1 \in N(i)$. Observe that $q_{j_1\text{RED}}$ is exactly the probability that after these T random walks, there exists a RED node, say j, such that count(j) > count(j') for all BLUE nodes j'. However, as $T \to \infty$,

$$\mathbb{P}[\text{for all BLUE books } j' \in V_2, \text{ count(j)} > \text{count(j')}] = \mathbb{P}[\text{for all BLUE books } j' \in V_2, \mathbb{P}[j_1 \xrightarrow{2} j] > \mathbb{P}[j_1 \xrightarrow{2} j']]$$

since as $T \to \infty$, count(j) $\to T \cdot \mathbb{P}[j_1 \xrightarrow{2} j]$ (by the Strong Law of Large Numbers). Therefore,

$$q_{j_1\text{RED}} = \mathbb{P}[\text{for all BLUE books } j' \in V_2, \mathbb{P}[j_1 \xrightarrow{2} j] > \mathbb{P}[j_1 \xrightarrow{2} j']]$$

Observe that for two RED books j_1 and j,

$$\mathbb{P}[j_1 \xrightarrow{2} j] = \sum_{i' \in N(j_1) \cap N(j)} \frac{1}{|N(j_1)|} \frac{1}{|N(i')|} = \sum_{i' \in V_1} \frac{Z_{i'j_1} Z_{i'j}}{|N(j_1)||N(i')|}$$

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By Lemma 6.1, in the limit as $n \to \infty$, with probability 1,

$$\mathbb{P}[j_1 \xrightarrow{2} j] \to \frac{1}{k} \frac{1}{mk/2n} \frac{mk^2(\frac{1}{4} + \mathrm{Var}(x_1))}{n^2} = \frac{1}{n}\left(\frac{1}{2} + 2\mathrm{Var}(x_1)\right)$$

Similarly, for a BLUE book j', in the limit as $n \to \infty$, with probability 1,

$$\mathbb{P}[j_1 \xrightarrow{2} j'] \to \frac{1}{k} \frac{1}{mk/2n} \frac{mk^2(\frac{1}{4} - \mathrm{Var}(x_1))}{n^2} = \frac{1}{n}\left(\frac{1}{2} - 2\mathrm{Var}(x_1)\right)$$

Since $\mathrm{Var}(x_1) > 0$, in the limit as $n \to \infty$, $\mathbb{P}[j_1 \xrightarrow{2} j] > \mathbb{P}[j_1 \xrightarrow{2} j']$ with probability 1. Therefore, $q_{j_1\mathrm{RED}} = 1$. By symmetry $q_{j_2\mathrm{RED}} = 1 - q_{j_2\mathrm{BLUE}} = 0$. Moreover, by Lemma 1, in the limit as $n \to \infty$, $\sum_{\substack{j_1 \in V_2 \\ j_1 \text{ is RED}}} \frac{Z_{ij_1}}{|N(i)|} = x_i$, with probability 1. Therefore, as $n \to \infty$,

$$p_r = x_i \tag{6.2}$$

The rest of the analysis is identical to Lemma 6.2. $\qquad\square$

This completes the proof of Theorem 6.

6.2 Proof of Theorem 5

Assume, without loss of generality, that $x_i > \frac{1}{2}$.

Let p_r be the probability that SimplePPR recommends a RED book to i. This probability is exactly equal to the probability that after T three-step random walks starting at i there exists a RED node, say j, such that such that $\mathtt{count(j)} > \mathtt{count(j')}$ for all BLUE nodes j'. However, as $T \to \infty$,

$$\mathbb{P}[\text{for all BLUE books } j' \in V_2, \mathtt{count(j)} > \mathtt{count(j')}] = \mathbb{P}[\text{for all BLUE books } j' \in V_2, \mathbb{P}[i \xrightarrow{3} j] > \mathbb{P}[i \xrightarrow{3} j']]$$

since as $T \to \infty$, $\mathtt{count(j)} \to T \cdot \mathbb{P}[i \xrightarrow{3} j]$ with probability 1 (by the Strong Law of Large Numbers). Therefore,

$$p_r = \mathbb{P}[\text{for all BLUE books } j' \in V_2, \ \mathbb{P}[i \xrightarrow{3} j] > \mathbb{P}[i \xrightarrow{3} j']]$$

For a RED book $j \in V_2$,

$$\mathbb{P}[i \xrightarrow{3} j] = \sum_{\substack{j_1 \in N(i) \\ j_1 \text{ is RED}}} \mathbb{P}[i \xrightarrow{1} j_1]\mathbb{P}[j_1 \xrightarrow{2} j] + \sum_{\substack{j_2 \in N(i) \\ j_2 \text{ is BLUE}}} \mathbb{P}[i \xrightarrow{1} j_2]\mathbb{P}[j_2 \xrightarrow{2} j]$$

$$\mathbb{P}[i \xrightarrow{3} j] = \sum_{\substack{j_1 \in N(i) \\ j_1 \text{ is RED}}} \frac{1}{|N(i)|}\mathbb{P}[j_1 \xrightarrow{2} j] + \sum_{\substack{j_2 \in N(i) \\ j_2 \text{ is BLUE}}} \frac{1}{|N(i)|}\mathbb{P}[j_2 \xrightarrow{2} j]$$

$$\mathbb{P}[i \xrightarrow{3} j] = \sum_{\substack{j_1 \in V_2 \\ j_1 \text{ is RED}}} \frac{Z_{ij_1}}{|N(i)|}\mathbb{P}[j_1 \xrightarrow{2} j] + \sum_{\substack{j_2 \in V_2 \\ j_2 \text{ is BLUE}}} \frac{Z_{ij_2}}{|N(i)|}\mathbb{P}[j_2 \xrightarrow{2} j]$$

As we showed in the proof of Lemma 6.3, in the limit as $n \to \infty$,

$$\mathbb{P}[j_1 \xrightarrow{2} j] \to \frac{1}{n}\left(\frac{1}{2} + 2\mathrm{Var}(x_1)\right) \text{ and (by symmetry) } \mathbb{P}[j_2 \xrightarrow{2} j] \to \frac{1}{n}\left(\frac{1}{2} - 2\mathrm{Var}(x_1)\right)$$

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with probability 1. Moreover, by Lemma 1, in the limit as $n \to \infty$, $\sum_{\substack{j_1 \in V_2 \\ j_1 \text{ is RED}}} \frac{Z_{ij_1}}{|N(i)|} \to x_i$, with probability 1. Therefore, with probability 1,

$$\mathbb{P}[i \xrightarrow{3} j] \to \frac{x_i}{n}\left(\frac{1}{2} + 2\mathrm{Var}(x_1)\right) + \frac{1-x_i}{n}\left(\frac{1}{2} - 2\mathrm{Var}(x_1)\right)$$

Similarly, for a BLUE book $j' \in V_2$, in the limit as $n \to \infty$, with probability 1,

$$\mathbb{P}[i \xrightarrow{3} j'] \to \frac{x_i}{n}\left(\frac{1}{2} - 2\mathrm{Var}(x_1)\right) + \frac{1-x_i}{n}\left(\frac{1}{2} + 2\mathrm{Var}(x_1)\right)$$

Since $x_i > \frac{1}{2}$ and $\mathrm{Var}(x_1) > 0$,

$$\mathbb{P}[i \xrightarrow{3} j] > \mathbb{P}[i \xrightarrow{3} j']$$

with probability 1. In other words, $p_r = 1$. Consider the general definition of a biased individual, where individual i accepts the recommendation of a RED book with probability x_i^b and accepts that of a BLUE book with probability $(1-x_i)^b$, where $b \geq 0$. Then, the probability that the recommended book was RED given that i accepted the recommendation is given by

$$\frac{p_r x_i^b}{p_r x_i^b + (1-x_i)^b(1-p_r)}$$

Since $p_r = 1$, the probability that a book recommended by SimplePPR was RED given that it was accepted is exactly p_r for all $b \geq 0$. Therefore, SimplePPR is polarizing for all $b \geq 0$.